Real Analogues of Pre-Schwarzian Univalence Criteria

M. Chuaqui
Facultad de Matemáticas
P. Universidad Católica de Chile
Casilla 306, Santiago 22 , CHILE

J. Gevirtz*
Facultad de Matemáticas
P. Universidad Católica de Chile
Casilla 306, Santiago 22, CHILE

Abstract

Let $Pf = f''/f'$ (the pre-Schwarzian) and $Sf = (Pf)' - \frac{1}{2}(Pf)^2$ (the Schwarzian). Because extremal behavior for Nehari's celebrated sharp univalence criteria of the form $|Sf(z)| \leq cp(|z|)$ in the unit disc $\mathbf{D}$ is manifested by functions which display extremal behavior for the corresponding univalence criteria for $f : (-1, 1) \to \mathbf{C}$, one is led to ask whether the same holds for the pre-Schwarzian analogues, $|Pf(z)| \leq cp(|z|)$, and in particular, for the longstanding open question as to the largest constant c for which $|Pf(z)| \leq c$ implies univalence in $\mathbf{D}$. For a wide class of functions p we apply a sequence of variational arguments to obtain a characterization of the largest c_p for which $|f''(x)/f'(x)| \leq c_p p(x)$ on $(-1, 1)$ implies that f is one-to-one as the first eigenvalue of a certain nonlinear boundary value problem of Sturm-Liouville type. Using this we show that for no such p is extremal behavior for a univalence criterion $|Pf(z)| \leq cp(|z|)$ manifested by an extremal function for a corresponding real criterion. In addition, we show that when $|f''(x)/f'(x)|$ is replaced by $||f''(x)||/||f'(x)||$ the numerically sharp real univalence criteria obtained extend, with the same constants, to mappings of the interval into $\mathbf{R^n}$.

Introduction and Preliminaries

The univalence criteria that have received the widest attention in the literature are those involving either the Schwarzian derivative $Sf = (f''/f')' - (1/2)(f''/f')^2$, or the operator $Pf = f''/f'$, which is sometimes referred to as the pre-Schwarzian of f. Indeed, it is probably not incorrect to affirm that the most celebrated *sharp* univalence criteria in the unit disc $\mathbf{D}$ are the original criteria of Nehari [8],

$$|Sf(z)| \leq \frac{2}{(1 - |z|^2)^2}\,, \tag{1}$$

$$|Sf(z)| \leq \frac{\pi^2}{2}\,, \tag{2}$$

and that of Becker [1]

$$|Pf(z)| \leq \frac{1}{1 - |z|^2}\,. \tag{3}$$

Criteria (1), (2) are particular instances of a broad class of univalence criteria, also due to Nehari [9], who showed that under suitable, but nonetheless very general, conditions (see Theorems 1 and 2 in [9]) on an analytic function p for which

$$(1 - x^2)^2 p(x) \text{ is nonincreasing on } [0, 1)\,, \tag{4}$$

*Both authors were partially supported by Fondecyt Grant # 1971055.
Key words: Pre-Schwarzian, univalence criteria, extremal function, real analogue.
1991 AMS Subject Classification #: 30C99, 30E99, 34A20.

the bound

$$|Sf(z)| \le C_p p(|z|) \tag{5}$$

is a sharp univalence criterion in $\mathbf{D}$, where C_p is the supremum of all numbers C such that the condition

$$|Sf(x)| \le Cp(|x|) \tag{6}$$

implies that $f : (-1, 1) \to \mathbf{C}$ is one-to-one, with a single function displaying extremal behavior for both (5) and (6) (see, e.g., [3]). This reduction to univalence in the context of functions of a real variable, which is a consequence of the fact that a function satisfying $Sf = 2g$ is the ratio of two linearly independent solutions of $u'' + gu = 0$, is the basis of Nehari's original proofs of (1) and (2), and in one way or the other underlies most of the theory of univalence criteria involving bounds on Sf. In this note we investigate the corresponding 'real' analogues of pre-Schwarzian univalence criteria. The analysis will make manifest (as is explained in the paragraph immediately following the corollary to Theorem 1 below) that in contrast to the Schwarzian case, there is no real variable phenomenon associated in this direct way with sharp criteria of the form $|Pf(z)| \le cp(|z|)$.

Throughout, $I = (-1, 1)$. For a given nonnegative function p on I we denote by $\mathcal{F}(p, c)$ the set of all nonconstant functions $f : I \to \mathbf{C}$ for which f' is locally Lipschitz continuous on I, and which satisfy $|f''(x)| \le cp(x)|f'(x)|$ a.e. there. We are interested in the supremum c_p of those numbers for which all $f \in \mathcal{F}(p, c)$ are one-to-one on I. To avoid possible considerations not germane to the essential issue we shall limit our discussion to *continuous* p. This means in particular that f' is nonvanishing on I for all $f \in \mathcal{F}(p, c)$. The treatment is based on the analysis of *extremal functions* for p, that is, functions $f \in \mathcal{F}(p, c_p)$ which are continuous on the closed interval $\bar{I}$, one-to-one on I and which in addition satisfy $f(-1) = f(1)$. As is made clear by a simple example (see the paragraph immediately following the proof of Proposition 4 below) such functions do not exist for every p of the indicated kind, but as we shall show, the additional requirement that

$$(x - t)p(x) \text{ is strictly increasing on } I\,, \tag{7}$$

for all $t \in \bar{I}$, is sufficient to insure their existence. This condition may be thought of as the pre-Schwarzian parallel of (4), and is assumed to hold in all that follows.

Proposition 1. *For all proper subintervals (a, b) of I*

$$(\frac{b-a}{2})p\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) < p(x) \tag{8}$$

for all $x \in I$. If, in addition, $[a, b] \subset I$ then there exist $c < 1, d > 0$ such that

$$(\frac{b-a}{2})p\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) < cp(x) - d$$

for all $x \in I$.

Proof: Let $\delta = (b-a)/2$ and write $(b+a)/2$ in the form $t(1-\delta)$, so that $t = (b+a)/(2-b+a)$, which is easily seen to lie in $\bar{I}$. Let $x \in I$. Then

$$(x-t)[p(x) - \delta p(\delta x + t(1-\delta))] = (x-t)p(x) - [\delta x + t(1-\delta) - t]p(\delta x + t(1-\delta))\,. \tag{9}$$

Assume for the moment that $x \ne t$. In light of (7), the sign of the left-hand side is the same as that of $x - (\delta x + t(1-\delta)) = (1-\delta)(x-t)$, which means that $p(x) - \delta p(\delta x + t(1-\delta))$ is positive, as desired. In the case $x = t$ the inequality follows since then $\delta x + t(1-\delta) = x$, and by assumption

$\delta < 1$. This establishes the first statement. For the second part, we note that (7) implies that $\lim_{x\to -1^+} p(x) = A$ and $\lim_{x\to 1^-} p(x) = B$ exist (with ∞ being possible values). If $B = \infty$, then obviously $\frac{b-a}{2} p(\frac{b-a}{2} 1 + \frac{b+a}{2}) < B$. Otherwise, p is continuous on $(-1, 1]$ and it follows from (7) that when $x = 1$ the right-hand side of (9) is positive, so that $\frac{b-a}{2} p(\frac{b-a}{2} 1 + \frac{b+a}{2}) < B$ in this case also. In an analogous way one sees that $\frac{b-a}{2} p(\frac{b-a}{2}(-1) + \frac{b+a}{2}) < A$. The desired conclusion follows from the continuity of $\frac{b-a}{2} p(\frac{b-a}{2} x + \frac{b+a}{2})$ on $\bar{I}$. □

We mention in passing that (7) and (8) are actually equivalent. For future use we also note that trivial calculations show that

$$p_\alpha(x) = \frac{1}{(1-x^2)^\alpha} \tag{10}$$

satisfies (7) for $0 \le \alpha \le 1$.

Proposition 2. *If $p(x) \le K/(1-x^2)$ then $c_p \ge \sqrt{2}/K$.*

Proof: Let $q(x) = \sqrt{2}/(1-x^2)$. We have to show that a nonconstant f satisfying $|f''(x)| \le q(x)|f'(x)|$ cannot take any value more than once on I. Since the operator Pf is not affected by affine changes in f, it is enough to show that $h = |f|^2$ cannot vanish twice. We have

$$h' = 2\Re\{\bar{f} f'\} \quad \text{and} \quad h'' = 2\Re\{\bar{f} f''\} + 2|f'|^2 ,$$

and taking into account the bound on $|f''|$, we see that

$$h'' \ge 2|f'|^2 - 2|f||f'|q = 2(|f'| - |f|q/2)^2 - |f|^2 q^2/2 \ge -|f|^2 q^2/2 = -(q^2/2)h .$$

Thus

$$h'' + \frac{h}{(1-x^2)^2} \ge 0 \quad \text{a.e. on } I .$$

From this it follows from the classical Sturm comparison theorem [5, p.375] that if h vanishes at distinct $a < b$ in I, then any nonzero y which satisfies $y'' + y/(1-x^2)^2 = 0$ and which vanishes at a must also vanish at some $b' \in (a, b]$. But this is impossible since all such y are of the form

$$y = C\sqrt{1-x^2}(\log\frac{1+x}{1-x} - \log\frac{1+a}{1-a}) ,$$

and so have a unique zero in I. □

Since by (7) $(1-x)p(x)$ and $(1+x)p(x)$ are decreasing and increasing, respectively, on I, it follows that

$$(1-x)p(x) \ge \lim_{x\to 1^-} (1-x)p(x) = L_p^+ \ge 0$$

and

$$(1+x)p(x) \ge \lim_{x\to -1^+} (1+x)p(x) = L_p^- \ge 0 ,$$

so that

$$p(x) \ge \max\{L_p^+/(1-x), L_p^-/(1+x)\} , \tag{11}$$

and

$$p(x) \sim L_p^+/(1-x) \; x \to 1^- \;\; , \;\; p(x) \sim L_p^-/(1+x) \; x \to -1^+ . \tag{12}$$

Let $L_p = \max\{L_p^+, L_p^-\}$.

Proposition 3. $0 < c_p < 1/L_p$ *(where 1/0=∞).*

Proof: It follows immediately from Proposition 2 and (12) that $c_p > 0$. For the upper bound, we first consider the case in which $L = L_p > 0$, and without loss of generality we may assume that $L = L_p^+$. Lines (11) and (12) imply that

$$M(c) = \int_{-1}^{1} \exp\left(c\left|\int_0^x p(t)dt\right|\right) dx \tag{13}$$

diverges for $c = 1/L$ and converges for $c < 1/L$. We construct an $f \in \mathcal{F}(p, 1/L)$ which is not one-to-one on I. Obviously, $\int_0^1 p(t)dt$ diverges. Let $f(-1) = 0$, and let $f'(x) = 1$ on $(-1, 0]$. Next, let $A \in (1/2, 1)$ satisfy

$$\exp\left(-\int_0^A p(t)dt/L\right) < \frac{1}{2}.$$

Then we extend f to $(-1, A]$ by letting $f''(x)/f'(x) = -p(x)/L$ on $[0, A)$. Next we let $B \in (A, 1)$ satisfy $\int_A^B p(t)dt/L = 3\pi/2$, and extend f to $(-1, B]$ by letting $f''(x)/f'(x) = ip(x)/L$ on $[A, B)$. Finally, on the remainder $[B, 1)$ of I we let $f''(x)/f'(x) = p(x)/L$. One sees that $f([-1, A))$ is an interval of the real axis of length greater than 1, that $f([A, B])$ is a curve of length less than 1/4, and that $\arg f'$ increases from 0 to $3\pi/2$ as x increases from A to B. Because $M(1/L)$ diverges, the image of the remainder of I is an infinite ray which moves vertically downward from $f(B)$. It is then clear that there exist $a \neq b$ in $(-1, 1)$ for which $f(a) = f(b)$. By continuity there must exist corresponding points if we replace $1/L$ throughout by a $c < 1/L$ sufficiently close to $1/L$. This shows that $c_p < 1/L$, as desired. Finally we consider the case in which $L_p = 0$. It follows easily from (7) that $p(x)$ has a positive lower bound m on I. Because $f(x) = e^{ikx}$, $k > \pi$ is not one-to-one on I, and $|f''/f'| = k \leq (k/m)p$, it follows that $c_p \leq k/m < \infty = 1/L$. $\square$

Proposition 4. *There exists an extremal function for p.*

Proof: The number $\delta = (\min\{1, 1/L_p - c_p\})/2 > 0$ by the preceding proposition. From the definition of c_p it follows that there is a sequence $\{g_n\}$ of functions in $\mathcal{F}(p, c_p + \delta/n)$, for which $g_n(0) = 0, g_n'(0) = 1$ and $g_n(a_n) = g_n(b_n)$, where $-1 < a_n < b_n < 1$. Proposition 1 tells us that by replacing $g_n(x)$ by

$$(\frac{2}{b_n - a_n})\left[g_n\left(\frac{b_n - a_n}{2}x + \frac{b_n + a_n}{2}\right) - g_n\left(\frac{b_n + a_n}{2}\right)\right]$$

we may assume that $a_n = -1$ and $b_n = 1$. Clearly, the sequence $\{g_n'\}$ is uniformly bounded and equicontinuous on each compact subinterval of I, so that by the Arzela-Ascoli theorem there is a subsequence $\{g_{n_i}'\}$ which converges locally uniformly to some function h in I. Let $g(x) = \int_0^x h(t)dt$. Since for $\alpha < \beta$ in I

$$|g_n'(\beta) - g_n'(\alpha)| = \left|\int_\alpha^\beta g_n''(x)dx\right| \leq \int_\alpha^\beta (c_p + \delta/n)p(x)|g_n'(x)|dx\,,$$

it follows that $g \in \mathcal{F}(p, c_p)$. Since $c_p + \delta/n < 1/L_p$ for all n, the sequence $\{g_n\}$ is equicontinuous on $\bar{I}$, so that g is continuous on $\bar{I}$. Obviously, $g(-1) = g(1)$. Finally, if g is not one-to-one on I, it can be replaced by an appropriate function of the form $g(\frac{b-a}{2}x + \frac{b+a}{2})$. $\square$

Remark. We stress that for there to exist an extremal function it is not sufficient that p be merely nonnegative and continuous on I. To see this, let $\delta > 0$ and p be the even function which vanishes on $[0, 1 - 2\delta]$, is linear on $(1 - 2\delta, 1 - \delta]$ and is equal to 1 on $(1 - \delta, 1]$. Then for small δ there is no extremal function for p, since in fact, $f(1) \neq f(-1)$ for all $f \in \mathcal{F}(p, c_p)$. Indeed, let

$f \in \mathcal{F}(p, c_p)$, and let $g = f/f'(0)$, so that $g'(0) = 1$. Then $g(1-2\delta) - g(2\delta - 1) = 2(1-2\delta)$, since $g'' = 0$ on $[2\delta - 1, 1 - 2\delta]$. The restriction to $[1-\delta, 1)$ of $e^{4\pi i x/\delta}$ shows that any function h for which $h''/h' = 4\pi i/\delta$ on $[1-\delta, 1)$ is not one-to-one on this interval, so that $c_p \leq 4\pi/\delta$. This means that for $x \in [0, 2\delta)$ our g satisfies $|g'(1-2\delta+x)| \leq e^{4\pi x/\delta}$, so that $|g(1) - g(1-2\delta)| \leq 2\delta e^{8\pi}$, and likewise for $|g(-1) - g(2\delta - 1)|$. Thus $|g(1) - g(-1)| \geq 2(1-2\delta) - 4\delta e^{8\pi} > 0$ for sufficiently small δ.

Main Results

We can now prove:

Theorem 1. *Let f be an extremal function for p. Then there exists a real number β such that*

$$\frac{f''(x)}{f'(x)} = c_p e^{i\beta} p(x) \frac{\overline{f(x)}}{|f(x)|}$$

for all $x \in I$.

Proof: We use a sequence of three variational arguments. Let $q = f''/f'$. Without loss of generality we may assume that $f'(0) = 1$, and that $f(-1) = f(1) = 0$, that is that

$$f(x) = \int_{-1}^{x} \exp\left(\int_0^y q(t)dt\right) dy\,.$$

We begin by showing that $|q(t)| = p(t)$ a.e. on I. Assume that this is not true, so that for some $\delta > 0$ there is a set $X \subset (-1+\delta, 1-\delta)$ of positive measure on which $|q(t)| < c_p p(t) - \delta$. Then for all measurable functions v with support in X for which the essential supremum of $|v|$ is at most δ, and all $z \in \mathbf{D}$ the functions of x given by

$$f(x; v, z) = \int_{-1}^{x} \exp\left(\int_0^y (q(t) + zv(t))dt\right) dy = \int_{-1}^{x} f'(y) \exp\left(z \int_0^y v(t)dt\right) dy$$

belong to $\mathcal{F}(p, c_p)$. For each such v, and each $x \in \bar{I}$, $f(x; v, z)$ is an analytic function of $z \in \mathbf{D}$. We have

$$\frac{\partial f}{\partial z}(x; v, 0) = \int_{-1}^{x} f'(y) \int_0^y v(t)dtdy = f(x) \int_0^x v(t)dt - \int_{-1}^{x} f(y)v(y)dy\,,$$

which is obviously continuous in x. If $\partial f(1; v, 0)/\partial z = A \neq 0$, then it follows from the argument principle that for $a', b' \in I$ sufficiently near -1 and 1, there will be some $z \in \mathbf{D}$ for which $f(a'; v, z) = f(b'; v, z)$. Let $-1 < a < a'$ and $b' < b < 1$, and let $g(x) = f(\frac{b-a}{2}x + \frac{b+a}{2}; v, z)$. Then g is not one-to-one on I. We have

$$\left|\frac{g''(x)}{g'(x)}\right| \leq c_p(\frac{b-a}{2})p\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) < cp(x)\,,$$

for some $c < c_p$ by the second part of Proposition 1. But this contradicts the definition of c_p since g is not one-to-one on I. Thus

$$\frac{\partial f(1; v, 0)}{\partial z} = -\int_{-1}^{1} f(y)v(y)dy = 0\,.$$

But since v can be any function with support in X for which the supremum of $|v|$ is at most δ, f must be identically zero on X. This contradicts the fact that f is an extremal function. Thus indeed $|q(t)| = c_p p(t)$ a.e. on I.

Hence there is a bounded measurable real-valued function φ such that $q = c_p p e^{i\varphi}$. For any C^∞-function ψ of compact support in I, the function

$$f(x;\psi,s) = \int_{-1}^{x} \exp\left(\int_0^y c_p p(t) e^{i(\varphi(t)+s\psi(t))} dt\right) dy$$

of x is in $\mathcal{F}(p, c_p)$ for all real s. Furthermore, one sees that

$$\frac{\partial f}{\partial s}(1;\psi,0) = i\int_{-1}^{1} f'(y) \int_0^y c_p p(t) e^{i\varphi(t)} \psi(t) dt dy$$

$$= -i\int_{-1}^{1} f(y) c_p p(y) e^{i\varphi(y)} \psi(y) dy\,,$$

since $f(-1) = f(1) = 0$.

Now, consider the set D of values of $\partial f(1;\psi,0)/\partial s$ as ψ ranges over the indicated class of functions. Clearly, D is a (real) linear subspace of $\mathbf{C}$. If D were all of $\mathbf{C}$ then a straightforward argument based on the implicit function theorem would show that for $a', b' \in I$ sufficiently near -1 and 1, there would be some ψ and some s for which $f(a';\psi,s) = f(b';\psi,s)$, and a verbatim repetition of the previous argument would produce a contradiction. Thus, D is either a line containing 0 or $\{0\}$; that is, there is a $\beta \in \mathbf{R}$ such that for all admissible ψ,

$$e^{-i\beta} \int_{-1}^{1} f(y) p(y) e^{i\varphi(y)} \psi(y) dy \in \mathbf{R}\,. \tag{14}$$

But this clearly means that

$$\rho = e^{-i\beta} c_p f p e^{i\varphi}$$

is a real-valued function on I.

Finally, we show that ρ does not change sign on I. Assume, to the contrary, that there are disjoint measurable sets S^+ and S^- whose closures are contained in I on which ρ is positive and negative, respectively. Let σ be a nonnegative bounded measurable function with support in $S^+ \cup S^-$. Then the function

$$f(x;\sigma,s) = \int_{-1}^{x} \exp\left(\int_0^y (1 - s\sigma(t)) q(t) dt\right) dy$$

$$= \int_{-1}^{x} f'(y) \exp\left(-s\int_0^y \sigma(t) q(t) dt\right) dy$$

of x is in $\mathcal{F}(p, c_p)$ for all sufficiently small $s > 0$. Since σ is a bounded function of compact support in I, and $f(1) = f(-1) = 0$,

$$f(1;\sigma,s) = \int_{-1}^{1} f'(y) \left(1 - s\int_0^y \sigma(t) q(t) dt + O(s^2)\right) dy$$

$$= s\int_{-1}^{1} f(y) q(y) \sigma(y) dy + O(s^2)\,.$$

But $fq = f c_p p e^{i\varphi} = e^{i\beta}\rho$, so that

$$\int_{-1}^{1} f(y) q(y) \sigma(y) dy = e^{i\beta} \int_{-1}^{1} \sigma(y) \rho(y) dy\,.$$

However, this last integral can be made to vanish for appropriate $\sigma \neq 0$ of the type described because of the differing signs of ρ on S^+ and S^-; for such σ, $f(1;\sigma,s)$ is therefore $O(s^2)$. Now fix such a σ and let $\delta > 0$ be such that $|(1-s\sigma(t))q(t)| < c_p p(t) - 2s\delta$ on a subset $X \subset S^+ \cup S^-$ of positive measure. Let w be a measurable function with support in X, satisfying $|w(t)| \leq 1$ there, and for which $A = \int_{-1}^{1} w(y)f(y)dy \neq 0$. Then we can apply our first variation to $f(x;\sigma,s)$ with $v = s\delta w$. That is, we consider for $z \in \mathbf{D}$

$$g(x;v,z) = \int_{-1}^{x} \exp\left(\int_0^y [(1-s\sigma(t))q(t) + zv(t)]\,dt\right) dy$$

$$= \int_{-1}^{x} f'(y;\sigma,s) \exp\left(z\int_0^y v(t)dt\right) dy$$

$$= \int_{1}^{x} f'(y;\sigma,s)\left(1 + z\int_0^y v(t)dy + \sum_{k=2}^{\infty}(\int_0^y v(t)dt)^k \frac{z^k}{k!}\right) dy\,.$$

Then this function is in $\mathcal{F}(p, c_p)$ and

$$g(1;v,z) = -B\delta s z + O(s^2) + O(s^2z^2)\,,$$

where $B = \int_{-1}^{1} f(y;\sigma,s)w(y)dy \to A$ as $s \to 0$. Again, a straightforward application of the argument principle will show that for sufficiently small s there is a $z \in \mathbf{D}$ for which $g(1;v,z) = 0 = g(-1;v,z)$. If $g(x;v,z)$ is one-to-one on I, then $g(x) = g(x;v,z) \leq c_p p(x) - s\delta < c_p p(x)$ on X contradicts the fact, established by our first variation, that an extremal function must satisfy $|g''/g'| = c_p p$ a.e. on I. Thus there is a proper subinterval (a,b) on which g is one-to-one, but such that $g(a) = g(b)$. But then in light of Proposition 1, $g(\frac{b-a}{2}x + \frac{b+a}{2})$ does not have this property of extremal functions. Thus indeed, $\rho = e^{-i\beta} f p e^{i\varphi}$ has no sign changes on I, and after replacing β by $\beta + \pi$, if necessary, we have $e^{-i\beta} f(x)p(x)e^{i\varphi(x)} = p(x)|f(x)|$ a.e. on I. Since $f''/f' = c_p p e^{i\varphi}$, we therefore have $f''/f' = c_p e^{i\beta} p|f|/f = c_p e^{i\beta} p\bar{f}/|f|$ as desired. □

Before continuing, we say a few words about the case in which p is an even function, that is, $p(x) = p(|x|)$. Let f be an extremal function for such a p, and let $y \in I$ be a point at which $|f|$ attains its maximum value. We claim that $y = 0$. If this is not the case, then we may assume without loss of generality that $y > 0$, and also that $f(y) > 0$, so that $f'(y)$ is pure imaginary. Let

$$g(x) = \begin{cases} f(y + (1-y)x)\,, & x \geq 0 \\ \bar{f}(y - (1-y)x)\,, & x < 0\,. \end{cases} \tag{15}$$

Then it is easy to see that g' is locally Lipschitz on I, since $f'(y)$ is pure imaginary. Furthermore, $|g''(x)| \leq c_p(1-y)p((1-y)x + y)|g'(x)|$, for $x > 0$, and by Proposition 1 this is strictly less than $c_p p(x)|g'(x)|$ (with $b = 1, a = 2y - 1$). For $x < 0$, $|g''(x)| \leq c_p(1-y)p((1-y)x - y)|g'(x)|$, and by Proposition 1 this is again strictly less than $c_p p(x)|g'(x)|$ (with $b = 1-2y, a = -1$). If, on the other hand, g is one-to-one, then it is an extremal function for p. But this contradicts Theorem 1. If it is not one-to-one then as we have seen, some function of the form $g(\frac{b-a}{2}x + \frac{b+a}{2})$ will be inconsistent with the theorem. Thus $y = 0$ and g, as defined, is an extremal function for p. Equations (15) simply say that $g(-x) = \overline{g(x)}$, and it follows from the uniqueness of solutions to ordinary differential equations that for even p, for any extremal function f there holds $f(-x)/f(0) = \overline{f(x)/f(0)}$.

Without loss of generality we may assume that the pure imaginary $f'(0)$ is ib, $b > 0$. One easily sees that $f'(-x) = -\overline{f'(x)}$, and $f''(-x) = \overline{f''(x)}$, from which it follows that $e^{i\beta}$ must be pure imaginary. Because the maximum value of $|f|$ on I is attained at $x = 0$, it follows that $\Re\{f''(0)\} = \Re\{e^{i\beta} c_p p(0) ib\}$, so that $e^{i\beta} = i$. Thus we have the following:

Corollary. *If p is even, then c_p is the smallest positive number c for which there exists a $b > 0$ such that the following boundary value problem on $[0,1]$ has a solution:*

$$f'' = icp\frac{\bar{f} f'}{|f|}\,,\ f(0) = 1\,, f'(0) = ib\,, f(1) = 0\,. \tag{16}$$

For given $b > 0$, $c > 0$, (16) with initial conditions $f(0) = 1, f'(0) = ib$ can be integrated numerically. In the special case that $p \equiv 1$, straightforward calculations of this sort suffice to show that the corresponding constant c_0, lies in the interval $(2.75, 2.80)$. It would be of interest to determine for which p, if any, one can integrate the corresponding system (16) explicitly, and so obtain a better description of the constant c_p. It also follows immediately from these considerations that, in strong contrast to extremal behavior for the Schwarzian criteria of Nehari alluded to in the opening paragraph, it *cannot ever happen* for even p that $|Pf(z)| \leq c_p p(|z|)$ is a sharp univalence criterion in $\mathbf{D}$ for which extremal behavior is manifested by an extremal function for the corresponding real criterion. To see this, assume that the extremal function $f_0(x) = r(x)e^{i\theta(x)}$ for the real criterion is the restriction to I of an analytic function in $\mathbf{D}$. As we have seen, we may normalize by assuming that $\theta(0) = 0, r(0) = 1$, and $f_0'(0) = i\theta'(0) = ib, b > 0$. Then $f_0''(x)/f_0'(x) = ic_p p(x)e^{-i\theta(x)}$, so that

$$(\frac{f_0''}{f_0'})'(0) = ic_p p'(0)e^{-i\theta(0)} + c_p p(0)e^{-i\theta(0)}\theta'(0) = c_p p(0) b\,,$$

since $p'(0) = 0$ because p is even. But then $f_0''(0)/f_0'(0)$ lies on the positive imaginary axis, whereas $(f_0''/f_0')'(0) > 0$, which means that for z near 0 on the positive imaginary axis, $|f_0''(z)/f_0'(z)| > |f_0''(|z|)/f_0'(|z|)| = c_p p(|z|)$.

We next show that Theorem 1 can be extended to mappings $F : I \to \mathbf{R^n}$. To do so, however, we need the following:

Proposition 5. *Let $D \subset \mathbf{R^2}$ be a domain and $X \subset \mathbf{R}$ be an interval. Let $G = h \circ g$, where $g : X \to D$ and $h : D \to \mathbf{R^n}$ are C^2. If h is an isometry then $||G'(x)|| = |g'(x)|$ and $||G''(x)|| \geq |g''(x)|$ on X.*

Proof: Let $g = (u_1, u_2)$ and $h = (w_1, ..., w_n)$. Then $G'(x) = J_h(g(x))g'(x)$, where J_h denotes the Jacobian matrix of h, that is, the matrix whose $(k,i)^{th}$ element is $\partial w_k/\partial x_i$, $1 \leq k \leq n, i = 1, 2$, and $g'(x)$ is the column vector with components $u_1'(x), u_2'(x)$. That h is an isometry means that the columns of J_h are orthonormal in D. From this, it follows immediately that $||G'(x)|| = |g'(x)|$. Calculation of G'' gives

$$G''(x) = J_h(g(x))g''(x) + Bg'(x)\,, \tag{17}$$

where B is the matrix whose $(k,i)^{th}$ element is

$$\frac{\partial^2 w_k}{\partial x_i^2}u_i' + \frac{\partial^2 w_k}{\partial x_i \partial x_j}u_j'\,,$$

where $\{i,j\} = \{1,2\}$. Thus the columns of $Bg'(x)$ are linear combinations of the three column vectors whose components are $\partial^2 w_k/\partial x_1^2, \partial^2 w_k/\partial x_1 \partial x_2$, and $\partial^2 w_k/\partial x_2^2$, respectively. But the orthonormality of the columns of J_h implies that these three column vectors are orthogonal to the columns of J_h. Equation (17) then says that $||G''(x)|| \geq ||J_h(g(x))g''(x)|| = |g''(x)|$. $\Box$

Theorem 2. *Let $F : I \to \mathbf{R^n}$, $n \geq 2$, be a nonconstant mapping with locally Lipschitz continuous derivative. If $||F''(x)|| \leq c_p p(x)||F'(x)||$ a.e. on I, then F is one-to-one on I.*

Proof: Let F satisfy the hypotheses. Assume that there are two points $a < b$ in I for which $F(a) = F(b)$. Then $H(x) = F(\frac{b-a}{2}x + \frac{b+a}{2})$ satisfies $H(-1) = H(1)$, and $||H''(x)|| \leq c_p \frac{b-a}{2} p(\frac{b-a}{2}x + \frac{b+a}{2})||H'(x)|| \leq (cp(x) - d)||H'(x)||$ for some $c < c_p$ and $d > 0$, by the second part of Proposition 1. Thus we can replace H by a C^∞-mapping $G : I \to \mathbf{R^n}$ with the following properties: $||G''(x)|| \leq cp(x)||G'(x)||, G(-1) = G(1) = 0$, G is one-to-one on I and furthermore that

$$G(x)\,,\; G'(x) \tag{18}$$

are linearly independent for all $x \in I$. Now let $X = X(s)$, $-s_0 \leq s \leq s_0$ be an arclength parametrization of the curve given by G on I. Furthermore, let $L(s) = ||X(s)||$. Then condition (18) simply says that $|L'(s)| < 1$ on $(-s_0, s_0)$. We now consider a curve in the plane given by $z = z(s) = L(s)e^{i\theta(s)}$, where the function θ is chosen so that $|z'(s)| = 1$, that is, so that

$$|L'(s) + iL(s)\theta'(s)| = 1\,.$$

The function

$$\theta(s) = \int_0^s \frac{\sqrt{1 - (L'(t))^2}}{L(t)}dt\,,\; -s_0 \leq s \leq s_0$$

has this property. Note that θ is strictly increasing. Now consider the mapping h, which, properly speaking, maps a covering surface of (part of) the plane onto a cone containing the curve $G(I)$: $h(rz(s)) = rX(s)$. It is well known and routinely verified that every single-valued branch of such a mapping is an isometry. If $s(x)$ is defined by $X(s(x)) = G(x)$, and $g(x) = z(s(x))$, $x \in I$, then by the preceding proposition we have that $|g'(x)| = ||G'(x)||$ and $|g''(x)| \leq ||G''(x)||$, so that $|g''(x)| \leq cp(x)|g'(x)|$ on I. Since $g(-1) = g(1)$, either g is itself an extremal function function for p or there is one of the form $g(\frac{b-a}{2}x + \frac{b+a}{2})$. In either case, since $c < c_p$ the conclusion of Theorem 1 is violated. This gives us the desired contradiction. □

In the special case $p \equiv 1$ Theorem 2 has a simple physical interpretation. Consider a ship propelled by an engine which obtains its fuel from the medium in which the ship travels. Assume that the medium is homogeneous, that the magnitude of the acceleration produced by the engine is directly proportional to the rate at which fuel is fed to it, and that the maximum rate at which fuel can be extracted is directly proportional to the ship's speed. The operators of the ship may, however, choose not to avail themselves of all the fuel that can be potentially gathered, and they are likewise free to control the direction in which the ship accelerates, by appropriately adjusting the direction in which a rocket points, for example. If $F(t), t \geq t_0$, gives the trajectory of the ship, then the above stipulations are equivalent to the condition $||F''(t)|| \leq K||F'(t)||$, where K is an appropriate constant. But then Theorem 2 gives a sharp lower bound on the time it takes the ship to return to any given point: if the ship is not at rest at time t_1, then it cannot return to position $F(t_1)$ until at least $2c_0/K$ units of time have elapsed, where, as above, c_0 is the constant corresponding to $p \equiv 1$.

Let us abbreviate by c_α the constant c_{p_α}, where p_α is an in (10). Similarly, we define γ_α to be the analytic counterpart of c_α, that is, γ_α is the supremum of all γ such that $|Pf(z)| \leq \gamma/(1-|z|^2)^\alpha$ implies that f is univalent in $\mathbf{D}$. As shown by Becker [1] and Becker and Pommerenke [2], $\gamma_1 = 1$. Although this is the only one of these γ_α whose value has actually been determined, Kudryashov [6] showed that $\gamma_0 \geq 3.03...$. On the other hand, as indicated above $c_0 \in (2.75, 2.80)$, and it follows from Proposition 2 that $c_1 \geq \sqrt{2}$. Thus $c_0 - \gamma_0 < 0 < c_1 - \gamma_1$. Since it is easy to see that $c_\alpha - \gamma_\alpha$ is continuous in α on $[0,1]$, we conlcude that there is an $\alpha \in [0,1]$ for which $|Pf(z)| \leq c_\alpha/(1-|z|^2)^\alpha$ is a sharp univalence criterion in $\mathbf{D}$. Some discussion of lower bounds for γ_α can be found in [4], [7] and [11].

We close with some comments about the determination of γ_0. Although this problem was considered as far back as 1955 (see [10], where it is shown that $\gamma_0 \geq \sqrt{6}$), so that it is at least almost as old as Nehari's original criteria, it nevertheless remains unsolved, the best lower bound to date being the above $\gamma_0 \geq 3.03...$. The functions $f(z) = e^{(1+\epsilon)\pi z}$, $\epsilon > 0$, show that $\gamma_0 \leq \pi$, and analogy with (2) together with simple intuition based on the relationship between $Pf(z)$ and the curvature of images of line segments, suggests that the value of γ_0 is, in fact, π. This conjecture is given quite explicit support by the fact that the functions $f_0(z) = Ae^{cz} + B$, where $|c| = \pi$, are local extrema for this problem in the following sense. On the one hand, they satisfy $|Pf_0(z)| = \pi$ in $\mathbf{D}$ and are univalent there but not in $\overline{\mathbf{D}}$, and on the other hand, there is an $\epsilon_0 > 0$ such that all other (nonconstant) functions for which $|Pf(z)| \leq \pi$ in $\mathbf{D}$ and $|Pf(0)| > \pi - \epsilon_0$ are univalent in the closed disk $\overline{\mathbf{D}}$ (see [4]). The determination of γ_0 is a challenging problem whose solution would constitute a significant addition to our still sparse stock of sharp univalence criteria.

References

[1] J. Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen, *J. Reine Angew. Math. 255 (1972), 23-43.*

[2] J. Becker and Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, *J. Reine Angew. Math. 354 (1984), 74-94.*

[3] M. Chuaqui and B. Osgood, An extension of a theorem of Gehring and Pommerenke, *Israel J. Math. 91 (1995), 393-407.*

[4] J. Gevirtz, On some radii of univalence and related univalence criteria, *Complex Variables Theory Appl. 4 (1984), 21-33.*

[5] E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley, 1969.

[6] S.N. Kudryashov, On some criteria for the univalence of analytic functions, *Math. Notes 13 (1973), 219-223.*

[7] K-W. Lau and M.C. Liu, On a criterion for the univalence of holomorphic functions, *Proc. Amer. Math. Soc. 80 (1980), 651-652.*

[8] Z. Nehari, The Schwarzian derivative and schlicht functions, *Bull. Amer. Math. Soc. 55 (1949), 545-551.*

[9] Z. Nehari, Univalence criteria depending on the Schwarzian derivative, *Illinois J. Math. 23 (1979), 345-351.*

[10] T. Umezawa, On the theory of univalent functions, *Tôhoku Math. J. (2) 7 (1955), 212-228.*

[11] S. Yamashita, On a theorem of Duren, Shapiro and Shields, *Proc. Amer. Math. Soc. 73 (1979), 180-182.*